Appendix A

Non-dimensionlizing Fluid Dynamic Variables

There are three fundamental units to describe physical quantities. These three units can be chosen as the units of length, time, and mass. Fluid dynamic equations can be represented in different unit system. Many text books present how to non-dimensionlize the fluid dynamic equations. But most times, the fundamental reasons for these choices are not given. For example, theoretically we can only use three fundamental units or independent flow variables to non-dimensionlize a physical system, but it seems that there are more than three independent quantities used in the non-dimensionlization of the fluid dynamic variables. Here, we will present how to non-dimensionlize fluid dynamic variables through a fundamental analysis.

For any fluid system, such as the NS or the Boltzmann equation, all physical quantities are measured in units. There are only three fundamental units as references to define a unit system. The equations can be considered as a physical law in such a unit frame. Different choices of units are equivalent to a transformation between different frames. Non-dimensionlization is not to make the physical quantity be dimensionless, but to take an appropriate unit system, where the equations related to the flow problem can be described conveniently.

Now let's consider a physical system described in two unit frames (l_0, t_0, m_0) and (l_1, t_1, m_1), such as $l_0 = 1m$ and $l_1 = 1mm$. Since in both unit frames, the same physical quantities are measured, where the measured length, time, and mass are given by

$$x(l_0) = x_*(l_1), t(t_0) = t_*(t_1), m(m_0) = m_*(m_1),$$

where the "times" is used between the value and unit. Here the values x, x_*, t, t_*, m and m_* are purely numbers, and the units are included in (l_0, t_0, m_0) and (l_1, t_1, m_1). Then, with the above connection, the specific values of length, time, and mass can be transformed from one unit frame to another one,

$$x = x_* \left(\frac{l_1}{l_0}\right), t = t_* \left(\frac{t_1}{t_0}\right), m = m_* \left(\frac{m_1}{m_0}\right).$$

Now let's study the transformation of flow variables between two unit frames. In the unit frame (0), the values of density, velocity and length can be expressed as ρ_∞, u_∞ and l_∞. Note that they are purely numbers in unit frame (0). In the unit frame (1), these numbers will be changed to $(\hat{\rho}, \hat{u}, \hat{l})$ with the connections,

$$\rho_\infty \left(\frac{m_0}{l_0^3}\right) = \hat{\rho} \left(\frac{m_1}{l_1^3}\right), u_\infty \left(\frac{l_0}{t_0}\right) = \hat{u} \left(\frac{l_1}{t_1}\right), l_\infty(l_0) = \hat{l}(l_1).$$

Obviously, for convenience, we can choose the units of unit frame (1) to have $\hat{\rho} = \hat{u} = \hat{l} = 1$. With the above requirement, i.e., any inflow condition around a flying vehicle becomes the same in frame (1), the connections between different unit frames are

$$l_1 = l_\infty l_0, t_1 = \frac{l_\infty}{u_\infty} t_0, m_1 = \rho_\infty l_\infty^3 m_0.$$

These are the basic rules to define the units for frame (1).

With the above choices, the values to describe the same length, time, and mass between two frames are connected by

$$x = l_\infty x_*, t = \frac{l_\infty}{u_\infty} t_*, m = \rho_\infty l_\infty^3 m_*.$$

Then, the values of flow variables can be changed from one unit system (0) to another one (1), such as

$$\rho \left(\frac{m_0}{l_0^3}\right) = \rho_* \left(\frac{m_1}{l_1^3}\right), p \left(\frac{m_0}{l_0 t_0^2}\right) = p_* \left(\frac{m_1}{l_1 t_1^2}\right), u \left(\frac{l_0}{t_0}\right) = u_* \left(\frac{l_1}{t_1}\right),$$

from which we have

$$\rho_* = \frac{\rho}{\rho_\infty}, p_* = \frac{p}{\rho_\infty u_\infty^2}, u_* = \frac{u}{u_\infty}.$$

Note that (ρ, p, u) and (ρ_*, p_*, u_*) are purely numbers in unit frames (0) and (1).

In terms of the dynamical viscosity coefficient, we have

$$\mu \left(\frac{m_0}{l_0 t_0}\right) = \mu_* \left(\frac{m_1}{l_1 t_1}\right),$$

and

$$\mu_* = \frac{\mu}{\rho_\infty l_\infty u_\infty}.$$

In unit frame (0), for a specific viscosity coefficient, such as the incoming flow with μ_∞, the value of corresponding viscosity coefficient in unit frame (1) becomes

$$\mu_{\infty,*} = \frac{\mu_\infty}{\rho_\infty l_\infty u_\infty} = \frac{1}{\mathrm{Re}},$$

where $\mathrm{Re} = \rho_\infty l_\infty u_\infty / \mu_\infty$. Then, for any other viscosity coefficient μ in unit frame (0), in frame (1) it becomes

$$\mu_* = \frac{1}{\mathrm{Re}} \left(\frac{\mu}{\mu_\infty} \right).$$

Similarly, the particle collision time has the unit of time, the transformation of collision times between two frames are

$$\tau_* = \tau \left(\frac{u_\infty}{l_\infty} \right).$$

Theoretically, there are only three independent units for a fluid dynamic system. Under this unit system, the molecular energy, which is a variable in a microscopic scale, can be expressed as a value on the order of 10^{-23} in the units of meter, second, and kilogram. Therefore, for the convenience purpose, a new unit temperature is introduced. For example, in unit frame (0), the unit of energy can be expressed as $m_0 l_0^2 / t_0^2$, which is defined again as kT_0, and k is a constant. The constant is more or less an exchange rate if we define two money system, such as US dollar and Chinese Yuan. Then, with the new unit temperature T_0 we have $l_0^2/t_0^2 = (k/m_0)T_0$, and k/m_0 can be defined as a new constant R.

Now we have two unit frames (0) and (1) with the units (l_0, t_0, m_0, T_0) and (l_1, t_1, m_1, T_1), where T_0 and T_1 are temperatures in two frames. Besides previous choices for the units transformation from $\rho_\infty, u_\infty, l_\infty$ to $\hat{\rho} = \hat{u} = \hat{l} = 1$, we now have an additional one,

$$T_\infty(T_0) = \hat{T}(T_1).$$

Similar, we can define the relation between T_0 and T_1 to make sure $\hat{T} = 1$. As a result, the temperature transformation between unit frames is

$$T_1 = T_\infty T_0.$$

Then, for any temperature T is unit frame (0), it becomes T_* in system (1),

$$T_* = T \frac{T_0}{T_1} = \frac{T}{T_\infty}.$$

Besides the values of flow variable transformation, the constants of exchange rates, such as k and R in frame (0), will have different values in frame (1). For example, based on the velocity square we have

$$RT_\infty \left(\frac{l_0^2}{t_0^2}\right) = \hat{R}\hat{T}\left(\frac{l_1^2}{t_1^2}\right),$$

from which we have

$$\hat{R} = RT_\infty \left(\frac{l_\infty^2}{u_\infty^2}\frac{1}{l_\infty^2}\right) = \frac{RT_\infty}{u_\infty^2} = \frac{1}{\gamma M_\infty^2},$$

where the reference Mach number is defined as $M_\infty = u_\infty/\sqrt{\gamma RT_\infty}$. Even though the universal constants can be different in different unit frames, within the same unit frame, such as frame (1), the value of the universal constant should keep the same value, such as

$$R_* = \hat{R} = 1/(\gamma M_\infty^2).$$

If k is the Boltzmann constant in unit frame (0), then the corresponding value k_* in unit frame (1) can be derived from

$$R_* = \frac{k_*}{m_*} = \frac{1}{\gamma M_\infty^2}.$$

Since

$$m_* = \frac{m}{\rho_\infty l_\infty^3},$$

we get

$$k_* = \frac{1}{\gamma M_\infty^2}\frac{m}{\rho_\infty l_\infty^3} = \frac{T_\infty}{\rho_\infty l_\infty^3 u_\infty^2}k.$$

After the definition of temperature, we can talk about the heat flux $-\kappa\nabla T$ and the heat conduction coefficient κ. In unit frame (0), the heat conduction coefficient κ is defined as

$$\kappa = \frac{\mu C_p}{\mathrm{Pr}} = \frac{\mu}{\mathrm{Pr}}\frac{\gamma}{\gamma-1}R,$$

where the specific heat capacity at constant pressure is defined as $C_p = \gamma R/(\gamma-1)$. In unit frame (1), it becomes

$$\kappa_* = \frac{\mu_*}{\mathrm{Pr}}\frac{\gamma}{\gamma-1}R_* = \frac{\mu}{\mu_\infty}\frac{1}{(\gamma-1)\mathrm{Pr}\mathrm{Re}M_\infty^2}.$$

In many cases, the above expression can be written as

$$\kappa_* = \tilde{\mu}\frac{1}{(\gamma-1)\mathrm{Pr}\mathrm{Re}M_\infty^2},$$

where $\tilde{\mu} = \mu/\mu_\infty$. Then, the heat flux in unit frame (1) is

$$\vec{q}_* = -\kappa_* \left(\frac{\partial T_*}{\partial x_*} + \frac{\partial T_*}{\partial y_*} + \frac{\partial T_*}{\partial z_*} \right),$$

and the stress terms, such as τ_{xy}, become

$$\tau_{*,xy} = \frac{1}{\mathrm{Re}} \left(\frac{\mu}{\mu_\infty} \right) \left(\frac{\partial u_*}{\partial y_*} + \frac{\partial v_*}{\partial x_*} \right).$$

All fluid dynamic equations, including the Boltzmann equation, can be expressed in the same way in unit frame (0) and unit frame (1), but with their own unit system. The connections between unit systems are based on the choices of $\rho_\infty, u_\infty, l_\infty, T_\infty$ in unit frame (0), which have the corresponding values $\hat{\rho} = \hat{u} = \hat{l} = \hat{T} = 1$ in unit frame (1). This brings great simplification in the computation in unit frame (1). This is only one of the choices, which is mainly used in applications of aerospace flow problems. Due to the freedom in choosing the units, sometimes it is much more convenient to use $k_B \to \hat{k} = 1, m \to \hat{m} = 1$, $u_\infty \to \hat{u} = 1$, and $l_\infty \to \hat{l} = 1$ to define a new unit system (1) in microflow applications. In the cosmology study, it may become easy to use the speed of light $\hat{c} = 1$, Planck constant $\hat{\hbar} = 1$, and the mass of the electron $\hat{m}_e = 1$ to define the unit frame (1).